The Program of the Winter Colloquium on Discrete Mathematics DSBA

At the beginning of the colloquium, you will take a ticket containing three tasks: a question on understanding definitions, a problem (like a typical problem from seminars and home assignments), and a theorem from the course to be proved. You will have about an hour to prepare your answer. You shall answer orally to one of the examiners, but you shall write your answer during the time for preparation.

The grade for the colloquium is formed as follows. You get your first point as soon as you come to the colloquium, another 2 points for the complete answer to the first question (on understanding the definitions), 3 points for the correct solution of the problem, and the last 4 points for the full proof.

According to the rules of HSE, in the case of detected cheating the student will get 0 points (for the whole colloquium).

1. The List of Definitions

A question on understanding definitions contains a definition (that you shall formulate) and a question on this definition that you shall answer. Example: «Definition of the preimage. Let $f(x) = x^2$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$. Find the preimage of the set $\{1, 2, 3, 4\}$.»

1. Logical operations: conjunction, disjunction, and negation.
2. Logical operations: implication, XOR, and equivalence.
3. Boolean function. Definition via truth tables and vector of values.
4. Dummy and significant variables of a boolean function.
5. Disjunctive normal form.
6. A set, a subset and equality of the sets.
7. Operations with sets: union, intersection, difference, and symmetrical difference. Euler-Venn diagrams.
8. De Morgan's laws and generalized form (for the arbitrary family of sets).
9. The law of contraposition.
10. Rule of sum.
11. Mathematical induction
12. Inclusion-exclusion principle
13. Rule of product.

14. Combinatorial numbers: number of permutations, number of k-subsets of an n-element set.
15. Characteristic function and its application for computation of cardinality of a finite set.
16. Functions. Domain and range.
17. Image and preimage (of sets).
18. Totally defined functions. Injections, surjections and bijections.
19. Undirected graph
20. Vertex degree (for undirected graphs), indegree and outdegree (for directed graphs)
21. Walk and closed walk
22. Subgraph
23. Connected component
24. Special graphs: path, cycle, and Boolean cube of rank n
25. Graph isomorphism (see CW 8)
26. Tree. Spanning tree (see CW 8).
27. Full binary tree of rank n (see HW 8)
28. Distance between two vertices. The diameter of a graph. (see CW 8)
29. Proper coloring of a graph
30. Directed graph
31. Strongly connected component
32. Eulerian walk

2. Examples of Problems

Problems from the tickets are similar to the problems below, but not all of them are listed below.

The list of the problems will be updated.

1. Construct a DNF-expansion for a boolean function defined by the vector 10010011.

2. Find the number of sequences of length k that consist from various elements of an n-element set.

3. Find the coefficient of a term x^3y^7 in the expansion of $(2x - y)^{10}$.

4. Find the coefficient of a term $x_1^3 x_2 x_4^5 x_5$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

5. Find the number of different words that could be constructed by permutation of letters of the word "MATHEMATICA".

6. Count the number of **a)** totally defined functions **b)** (partially defined) functions **c)** injections from a 5-element set to a 9 element set.

7. A function f is defined from a set $\{1, 2, \dots, 8\}$ to a set $\{a, b, \dots, e\}$ as follows:

$$f : 1 \mapsto a, \quad 2 \mapsto a, \quad 3 \mapsto c, \quad 4 \mapsto d, \quad 5 \mapsto c, \quad 7 \mapsto d.$$

Find **a)** $\mathrm{Dom}(f)$ **b)** $\mathrm{Range}(f)$ **c)** $f(\{1, 2, 3\})$ **d)** $f^{-1}(c)$ **e)** $f(\{1, 2, 3, 5, 6\})$ **f)** $f^{-1}(\{a, b, c\})$

8. Prove that there is no graph with five vertices with degrees 4, 4, 4, 4, 2.

9. Prove that any connected graph has a spanning tree.

3. The List of Theorems

1. Each boolean function has a DNF expansion.
2. Generalized De Morgan's laws
3. Inclusion-exclusion principle
4. The number of k-subsets of an n-element set is $\binom{n}{k}$.
5. Binomial theorem.
6. Properties of Pascal's triangle: symmetry of coefficients in a line, elements in a line increase from the borders to the center.
7. **Edited:** Properties of Pascal's triangle: the sum of elements in a line, the lower bound for the central coefficient:
$$\binom{2n}{n} \geqslant \frac{2^{2n}}{2n+1}$$
8. The number of solutions of equation $x_1 + x_2 + \ldots + x_k = n$ in non-negative integers.
9. Multinomial theorem.
10. **Edited:** The sum of all the vertices' degrees in a graph is $2 \times |E|$. The number of people in this room that know exactly odd number of people (in this room) is even (acquaintance is mutual).
11. **Edited:** The number of connected components is at least $|V| - |E|$. If G is a connected graph such that $|E| = |V| - 1$ then deleting of any edge makes G disconnected.
12. For any 6 people in this room the following statement holds. If no 3 of them know each other then at least 3 of them do not know each other (acquaintance is mutual).
13. If G is a connected graph such that deleting of each edge makes G disconnected then G has no cycles.
14. If G is a connected graph that has no cycles then every two vertices of G are connected by the unique path.
15. If every two vertices of G are connected by the unique path then $|E| = |V| - 1$.
16. The criterion of 2-colorability: a graph is 2-colorable if and only if it has no closed-walk of an odd length.
17. A graph has a cycle of an odd length if and only if it has a closed-walk of an odd length.
18. The following conditions are equivalent for a directed graph
    (a) each strongly connected component consists of exactly one vertex
    (b) the graph is acyclic
    (c) the vertices can be enumerated so that $(u, v) \in E$ only if $\#_u < \#_v$ (an edge goes only from a vertex with a smaller number to a vertex with a greater number).
19. An undirected connected graph has an Eulerian closed-walk if and only if each vertex has an even degree.