Discrete Mathematics **DSBA**

Seminar 3. Sets and combinatorics

1. There is only one student in the group who have studied C++, Java, Python and Haskell. Each 3 languages from the list have been studied by two students, each 2 languages have been studied by 6 students. Each language have been studied by 15 students. What is the minimal possible number of students in the group?

2. Compute the value of the boolean function for each assignment of the variables:

$$x_1 \oplus x_2 \oplus x_3 \oplus (x_1 \wedge x_2) \oplus (x_2 \wedge x_3) \oplus (x_3 \wedge x_1) \oplus (x_1 \wedge x_2 \wedge x_3).$$

Quiz: *How the result is related to the inclusion-exclusion formula?*

3. Prove that the number of binary words of length n such that each word contain exactly k ones is the same as the number of subsets of the set $\{1, 2, \ldots, n\}$ of size k.

4. Let A and B be arbitrary sets (not necessary finite). Prove that the following properties are equivalent: "There exists an injection $f\colon A \to B$" and "There exists a surjection $g\colon B \to A$".

5. Which number is greater: the number of partitions of a 20-element set to 6 (nonempty) sets or the number of its subsets of size 5?

6. A *partition* of a positive integer N in k parts is a sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = N$. Which number is greater: the number of partitions of N with at most k parts or the number of partitions of $N + k$ with exactly k parts?

Remark: 2^A *denotes the set of all subsets of the set A. A set consisting of the subsets of set S is called a* family *of subsets S.*

7. A function f maps a family $\mathcal{P} \subseteq 2^{\{1,2,\ldots,n\}}$ to a Zhegalkin polynomial P as follows:

$$P = \bigoplus_{S \in \mathcal{P}} \bigwedge_{i \in S} x_i,$$

I.e. if $\mathcal{P} = \{\varnothing, \{1\}, \{1, 2\}, \{2, 3, 4\}\}$, then $f(\mathcal{P}) = 1 \oplus x_1 \oplus x_1x_2 \oplus x_2x_3x_4$ (we assume here that a conjunction of elements from the empty set equals 1).

Prove that **a)** f is a bijection; **b)** $f(\mathcal{P} \triangle \mathcal{Q}) = f(\mathcal{P}) \oplus f(\mathcal{Q})$.

A function g maps a Zhegalkin polinomial P to the function f_P which P computes. Prove that **c)** g is injection; **d)** g is surjection (see problem 6 of the week 1 about the Shannon expansion).

e) Prove that the number of boolean functions of n variables is exactly the same as the number of families of subsets of the set $\{1, 2, \ldots, n\}$.

8. A an B are finite sets such that there exist an injection $f\colon A \to B$ and a surjection $g\colon A \to B$. Prove that in this case also exists a bijection $h\colon A \to B$.

9. Which number is greater: the number of partitions of N with at most k parts or the number of partitions of N with parts of size at mot k?

10*. Which number is greater: the number of injections from a 5-element set to a 20-element set or the number of surjections from a 20-element set to a 5-element set?

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Home Assignment 3

1. How many numbers from 1 to 1000 are not devisable neither by 3 nor by 5 nor by 7?

2. Each cell of a 3×4 table could be either colored in black or left blank. Count the number of colorings of the table such that among blank cells there is the first raw or the last raw, or two middle columns (both simultaneously).

3. A group of people is wanted for a flight to Mars. The group should satisfy the following requirements. Everyone must own at least one of the professions of a cook, a medic, a pilot or an astronomer. At the same time, each profession from the list should be owned by exactly 6 people in the group. In addition, there should be exactly one person in the group who owns all these professions; each pair of professions must be owned by exactly 4 people; each triple by exactly 2.

Is it possible to satisfy these requirements?

4. Prove that the following proposition holds for each two nonempty finite sets A and B: "If there is no injection from A to B then there exists a surjection from A to B."

5. Construct a bijection between finite subsets of the set of positive integers and (strictly) increasing finite sequences of positive integers.

6. Which number is greater (answer for all positive n and k): the number of partitions of a n-element set with at most k sets[1] or the number of partitions of a $(n+k)$-element set with exactly k sets?

[1] There is no empty set in the partition. If A and B are sets from a partition then $A \cap B = \varnothing$.