Discrete Mathematics **DSBA**

Seminar 7. Graphs I.

On terminology. We assume that graphs have no loops and no multiple edges (such graphs are known as *simple*). A graph A is a *subgraph* of a graph B if A can be constructed by deleting edges and vertices (with all the adjacent edges) from B.

A *graph path* P_n has n vertices $v_1, \dots v_n$ and edges $\{v_i, v_{i+1}\}$ $(1 \leqslant i \leqslant n-1)$. So it has $n-1$ edges it is the *length* of the path.

1. Prove that there is no graph with five vertices with degrees 4, 4, 4, 4, 2.

2. A graph G has 100 vertices and 800 edges.

a) Prove that G has at least one vertex with degree at least 16.

b) Is it possible that each vertex of G has degree 16?

3. A n-dimensional *Boolean cube* is a graph B_n defined as follows. The vertices of B_n are the binary words of length n; two vertices are adjacent if the words are differs in exactly one position. For example, the neighbours of 1101 are 0101, 1001, 1111, 1100.

a) How many vertices does B_n have?

b) How many edges does B_n have?

c) How many subgraphs that are the paths of length 2 (graphs P_3) does B_n have?

d) Is it true, that a graph B_3 has a walk of length 3000?

e) Is it true, that B_3 has a (simple) path of length 8? of length 7?

Graphs $G = \langle V, E\rangle$ и $G' = \langle V', E'\rangle$ are *isomorphic* if there exists a bijection $f : V \to V'$ such that $\{u, v\} \in E \iff \{f(u), f(v)\} \in E'$.

4. A graph $S_n = \langle V, E\rangle$ consist vertices $V = 2^{\{1,2,\dots,n\}}$ (a vertex $v \in V$ is a subset of the set $\{1, 2, 3, \dots n\}$); vertices v and u are adjacent iff $|u \bigtriangleup v| = 1$.

a) Prove that S_n is isomorphic to B_n.

b) How many tuples of different subsets $A_1, A_2, A_3 \subseteq \{1, 2, \dots, n\}$ are there such that $|A_1 \triangle A_2| = |A_2 \triangle A_3| = 1$?

A *complement* $\bar{G}$ of a graph G is a graph with the same vertices as G's vertices such that $\bar{G}$ has an edge $\{u, v\}$ iff G hasn't.

5. Prove that a graph or its compliment are connected (may be both).

6. What is the maximum number of edges that can have a disconnected (not connected) graph with n vertices?

7. It is known that a graph has 1000 vertices and 2017 edges. Is it true that such a graph can have no (simple) paths of length 64?

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Home Assignment 7

1. What is the largest number of edges a graph with 10 nodes can have?

2. Is there a graph on 8 vertices with 23 edges and a vertex of degree 1?

3. Find all the graphs such that each pair of edges has a common vertex.

4. How many subgraphs does an edgeless graph on n nodes have? How many subgraphs does a triangle (C_3) have?

5. A graph has 400 vertices, each vertex has degree 201. Prove that there is a subgraph in this graph that is a cycle of length 3 (graph C_3).

6. Arithmetics is a marvelous country with 9 cities named 1, 2, 3, 4, 5, 6, 7, 8, 9. A traveler discovered that two cities A and B are connected by an airline if and only if a two-digit number AB is divisible by 3. Is it possible to go from the city 1 to city 9 using these airlines (possibly with transfers)?

7. There are 15 cities in a marvelous country and each city is connected by roads with at least 7 others. Prove that a traveler can go from each city to each another by the roads (possibly with transfers).

8. A graph G has the set of vertices $V = \{1, 2, 3, 5, 6, 10, 15, 30\}$. G has an edge $\{u, v\}$ (let $u < v$) if and only if the following conditions hold:

- v is divisible by u;
- there is no vertex $s \in V, s \neq v$ such that v is divisible by s and s is divisible by u.

a) Draw the graph G.

b) Is this graph isomorphic to the Boolean cube B_3? If the answer is positive, construct the bijection.