Discrete Mathematics **DSBA**

Seminar 8. Graphs II. Trees

On terminology. We assume that graphs have no loops and no multiple edges (such graphs are known as *simple*). A graph A is a *subgraph* of a graph B if A can be constructed by deleting edges and vertices (with all the adjacent edges) from B.

A *tree* is a connected graph without cycles.

A graph coloring is *proper* if the endpoints of each edge has different colors.

Graphs $G = \langle V, E\rangle$ и $G' = \langle V', E'\rangle$ are *isomorphic* if there exists a bijection $f : V \to V'$ such that $\{u, v\} \in E \iff \{f(u), f(v)\} \in E'$.

A n-dimensional *Boolean cube* is a graph B_n defined as follows. The vertices of B_n are the binary words of length n; two vertices are adjacent if the words are differs in exactly one position. For example, the neighbours of 1101 are 0101, 1001, 1111, 1100.

A distance $\rho(u, v)$ between two vertices u and v in a connected graph is the length of the shortest path between the vertices u and v. A *diameter* of a graph G is a number $d_G = \max_{u,v\in V} \rho(u, v)$ or a path of length d_G (G may have many diameters).

1. A tree has 2018 vertices. Does it contain a path of length 3?

2. Does there exist a tree on 9 vertices with 2 vertices of degree 5?

3. A tree has no vertices of degree 2. Prove that the number of external vertices (vertices of degree 1) is at least a half of the total number of vertices.

A spanning tree is a subgraph of a graph that is a tree on all the vertices of the graph.

4. Prove that any connected graph has a spanning tree.

5. a) Find the diameter of a Boolean cube B_n. Answer for $n = 3$ at first.

b) Construct a spanning tree of a graph B_3.

A rooted tree is a tree in which one vertex has been designated the root. A *leaf* of a rooted tree is a vertex of degree 1 that is not a root. A *depth* of a rooted tree is the length of the longest path started in the root.

c) Prove that the minimal (by all the rooted spanning trees) depth of a spanning tree of a graph is at least a half of the diameter of the graph.

6. a) Prove that an arbitrary tree is 2-colorable. **b)** How many distinct proper colorings of a tree are there?

7. Prove that in a tree on $2n$ vertices you can choose n vertices so that no pair of chosen vertices is connected by an edge (such sets of vertices are called *independent sets*).

8. *A clique* of size n in a graph G is a subgraph of G isomorphic to the complete graph K_n (each vertices of a complete graph are adjacent).

a) Prove that a graph G contains a clique of size n if and only if its complement $\bar{G}$ contains an independent set on n vertices.

b) Prove that if a graph G contains a clique of size n then its vertices cannot be proper colored in $n - 1$ colors.

9. Let G be a connected graph. Prove that you can choose a vertex of G so that after its deleting (with all the adjacent edges) G remains connected.

Discrete Mathematics **DSBA**

Home Assignment 8

A *full binary tree* of rank n is defined as follows. The vertices are the binary words of length at most n (including the *empty word* that is a word of length 0). Two words u and v (let u be shorter than v) are adjacent if and only if $v = u0$ or $v = u1$.

1. The degree of each vertex of a graph G is 2. Is it true that G is 2-colorable?

2. How many proper colorings of a graph P_n (path on n vertices of length $n-1$) in red, blue and green are there exist?

3. In a tree on 2018 vertices, exactly three vertices have degree 1. How many vertices have degree 3?

4. There are two trees on n vertices, each has a diameter of length d. Is it possible to add an edge between some two vertices from different trees so that the diameter of the resulting tree would be d?

5. A non-2-colorable graph is called minimal if, after removing any edge, it becomes 2-colorable. Prove that there is at least one isolated vertex (a vertex of degree 0) on a minimal non 2-colored graph on 1000 vertices (that is, a vertex of degree 0).

6. Find the number of diameters (pathes) in a full binary tree of rank n.

7. Prove that a Boolean cube B_{2n} has a subgraph isomorphic to a full binary tree of rank n.

8. Does a Boolean cube have a spanning tree in which all but two vertices have degree 2?