Discrete Mathematics **DSBA**

Seminar 10. Binary Relations

Notation. xPy stands for $(x, y) \in P$. $P^{\intercal}$ is a transpose (or converse) relation that consists of pairs (y, x) such that $(x, y) \in P$. $\overline{P}$ is a complement to the relation P, it consists of all the pairs that do not belong to P. $Q \circ P \subseteq X \times Z$ is a composition of relations $P \subseteq X \times Y$ and $Q \subseteq Y \times Z$. It is defined by the formula

$$Q \circ P = \{(x, z) \mid \exists y \in Y : (xPy) \wedge (yQz)\}.$$

Note that the order of operands in the composition is chosen so that a composition of functions $f \circ g$ is a function $f(g(x))$.

1. Draw a bipartite graph corresponding to the binary relation $R \subseteq \{a, b, c, d, e\} \times \{1, 2, 3, 4, 5, 6\}$:

$$R = \{(a, 1), (a, 2), (b, 4), (c, 3), (d, 5)\}.$$

and answer the following questions.

a) Is R a function?

b) Is $R^{\intercal}$ is a function?

2. Answer the following questions for the binary relation $R \subseteq \{1, 2, 3\} \times \{1, 2, 3\}$. Is R reflexive? symmetric? transitive? an equivalence relation? For each relation R draw the corresponding graph. Choose undirected graphs for symmetric relations, in the case of non-reflexive relation use loops.

a) $R = \{(1, 2), (2, 3), (1, 3)\}$

b) $R = \{(1, 2), (2, 1), (1, 1)\}$

c) $R = \{(1, 1), (2, 2), (3, 3)\}$

d) $R = \{(x, y) \mid x, y \in \{1, 2, 3\}\}$

e) $R = \varnothing$

3. Compute the resulting binary relation. Describe it via math. symbols or in English. Each relation is defined over real numbers.

a) $\overline{(>)}$; **b)** $(>)^{\intercal}$; **c)** $(\geq)\Delta(\leq)$; **d)** $(>) \cap (<)$; **e)** $(=) \circ (>)$; **f)** $(<) \circ (<)$; **g)** $(<) \circ (>)$.

4. Are the following binary relations defined on th set of points or the set of lines of a geometrical plain ($\mathbb{R}^2$) reflexive, symmetric, transitive?

a) aPb = "Points a and b lie on a line"

b) aQb = "Line a is perpendicular to line b"

c) aRb = "A line a is parallel to a line b" (The answer depends on your textbook on geometry)

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Home Assignment 10

1. Answer the following questions for the binary relation $R \subseteq \{1,2,3\} \times \{1,2,3\}$. Is R reflexive? symmetric? transitive? an equivalence relation? For each relation R draw the corresponding graph. Choose undirected graphs for symmetric relations, in the case of non-reflexive relation use loops.

a) $R = \{(1,1),(2,2),(3,3),(1,2),(1,3),(3,2)\}$

b) $R = \{(1,1),(1,2),(2,1),(2,2)\}$

2. Express the relation "Nephew" via the relations "Mother" and "Father" and arbitrary operations with relations. Formally, xNy holds if and only if x is a nephew of y, xMy (xFy) holds if and only if x is the mother (father) of y.

3. Is it true that for arbitrary transitive relations $P_1, P_2 \subseteq A \times A$ the following relations would be transitive?

a) $\overline{P_1}$ **b)** $P_1 \cap P_2$

4. Is it true that for arbitrary transitive relations $P_1, P_2 \subseteq A \times A$ the following relations would be transitive?
a) $P_1 \cup P_2$ **b)** $P_1 \circ P_2$

5. A binary relation on the set of 6 elements contains 33 pairs. Can it be **a)** symmetric **b)** transitive?

6. Which of the following relations are equivalence relations on $\mathbb{N}$?

a) xPy : numbers x and y have the same last digit (hereinafter in decimal representation)

b) xQy : numbers x and y differs in the exactly one digit.

c) xRy : the difference between the sums of digits S_x and S_y is even. Formally, let $\overline{x_n x_{n-1} \ldots x_1 x_0}$ be the decimal representation of x; $S_x = \sum\limits_{k=0}^{n} x_k$.

7. How many relations R on the set $\{1,2,3,4\}$ are the equivalence relations?

8. Let R be an equivalence relation on a set A. Prove that there exist such a set B and a mapping $f : A \to B$ such that each equivalence class C can be expressed as $C = f^{-1}(b)$ for some element $b \in B$ (here f^{-1} is the preimage).