Discrete Mathematics **DSBA**

Seminar 11. Integers, Divisors and Primes

1. It is known that a, b, c, d are positive integers, $ab = cd$ and a is divisible by c. Prove that d is divisible by b.

2. It is known that the quotient of the division with the remainder of a positive integer m by 13 is the same as the quotient of the division of m by 15. In the first case the remainder is 8 and in the second case it is 0. Find m.

3. Find the remainder of the devision

a) 100^{100} by 99

b) $\binom{15}{8}$ by 13;

c) $20^2 + 21^2 + 22^2$ by 23;

d) $\binom{32}{3}$ by 33;

e) $8^{8^{8^8}}$ by 13.

4. Formulate and prove divisibility rules for numbers **a)** 9 **b)** 11 (in the decimal representation).

5. a) Which digits occur on the last position of numbers that are the powers of 3? **b)** Prove that the digit on the last but one position of a power of 3 is even.

6. a) Let p be a prime number greater than 3. Prove that $p^2 - 1$ is divisible by 24. **b)** Prove that for any integer a the number $a^{73} - a$ is divisible by 2, by 3, by 5, by 7, by 13, by 19, by 37, by 73.

7. Prove that $(p-1)!$ gives the remainder -1 modulo p for any prime number p.

8. Prove that for any positive integer $n \geqslant 2$ there is a prime number between n and $n!$.

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Home Assignment 11

1. Find the last two digits of the number 99^{1000}.

2. Which of the following assertions about the integers a, b, c are true: (1) if a is divisible by c, and b is not divisible by c, then $a+b$ is not divisible by c; (2) if a is not divisible by c and b is not divisible by c, then $a+b$ is not divisible by c; (3) if a is not divisible by c and b is not divisible by c, then ab is not divisible by c; (4) if a is divisible by b and b is divisible by c, then ab is divisible by c^2? Prove the assertions that are true and give counterexamples to the false ones.

The correct answer (with the proof) to each question gives 1 point to the overall grade for the task (maximum 4 points).

3. Let x and y be integers. Prove that the number $x+10y$ is divisible by 13 if and only if $y+4x$ is divisible by 13.

4. A positive integer a is even, but not divisible by 4. Show that the number of (positive) even divisors of a is equal to the number of (positive) odd divisors of a.

5. A positive integer x consists of the following digits: 100 zeros, 100 ones, and 100 twos. Can x be a perfect square ($x=n^2$)?

6. Prove that the numbers a^2 and b^2 have the same remainder after division by $a-b$ if a and b are positive integers and $a>b$.

7. Find the smallest positive integer N such that both the sum of the digits of the number N and the sum of the digits of the number $N+1$ are divisible by 7.

8. It is known that the number $a^{10}+b^{10}+c^{10}+d^{10}+e^{10}+f^{10}$ is divisible by 11. Prove that the number $abcdef$ (the product of a, b, c, d, e, f) is divisible by 11^6. Here a, b, c, d, e, f are integers.