Seminar 17. Probability III. Expectations of Random Variables

1. Two dice have been thrown. Find the expected value of the sum of points on them.

2. It is known about non-negative random variable X that $\Pr[X < 5] = 1/2$ and $\Pr[X > 5] = 1/2$. Find all possible values of the expectation $\mathrm{E}[X]$.

3. Tom Sawyer paints the fence, consisting of 20 boards. He paints the first board and then he paints each next board with probability 4/5 and with probability 1/5 he goes swimming (and stops painting). Find the expected number of painted boards. (Note he will paint at least one board.)

Recommendation. Solve the problem in two ways: using only the definition of expectation and using the linearity property (decompose the random variable into the sum of the auxiliary random variables).

4. Each element of an n-element set independently of the others is included in the set S_p with probability p. Find the expectation of the number of elements in the set S_p.

5. A random ten-element subset S of integers from 0 to 29 is chosen. Find the expected value of the sum of the numbers in S.

6. A student for doing homework gets a mark from 1 to 10. The average mark for a series of homework is 6. Prove that the rate of number of homeworks for which the score is less than 4 to the number of all homeworks does not exceed 4/7.

Recommendation. Use Chebyshev's inequality for a modified random variable.

7. 15 boys and 15 girls are form a random queue. How many girls on average stays earlier than all boys?

8. (Chebyshev's inequality.) For a random variable X we denote $M = \mathrm{E}[X]$, $D = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$. Prove that

$$\Pr\left[|X - M| \geqslant a\right] \leqslant \frac{D}{a^2} \, .$$

9. A uniform probabilistic space consists of binary words of length n. Prove that the probability of the event «number of ones per word differ from $n/2$ by no less than $\sqrt{n}$ » not exceeds 1/4.

Discrete Mathematics **DSBA**

Home Assignment 17

1. A player plays in the casino in the following game. Makes a bet c pounds, says the dealer a number from 1 to 6, then rolls three dices. If his number does not fall out, the player gets nothing. If the number falls, the player's payoff is $k \times c$ pounds where k is the number of dices with player's number. So, if the player bets a hundred of pounds and his number appears exactly twice, the player gets 200 pounds, and if the number doesn't appear, then his payoff is -100 pounds. Find the expectation of a player's payoff if he bets 100 pounds.

2. In the lottery winning payoffs take 40% of the value of sold tickets. Each ticket costs 100 rubles. Prove that probability of winning 5000 rubles (or more) less than 1%.

3. According to the mortality tables for 1693 the average life expectancy was 26 years. At the same time the probability of living no more than 8 years was 1/2. What was the average lifespan for those people who have lived at least 8 years? (Specify possible values under given conditions, only full years are taken into account.)

4. A random word of length 20 over alphabet $\{a, b\}$ is chosen. Find the expectation of the number of subwords ab in this word (ab is a subword of w if a stands on a place k and b stands on the place $k + 1$).

5. An *inversion* in a permutation $(a_1, a_2, \dots, a_n)$ of numbers $1, \dots, n$ is a pair of indices $(i.j)$ such that $i < j$ and $a_i > a_j$. Let π be a random permutation (the distribution is uniform). Find the expectation $\mathrm{E}[I(\pi)]$ of the number of inversions $I(\pi)$.

6. Let X be a non-negative random variable. It is known that $\mathrm{E}[2^X] = 5$. Prove that

$$\Pr[X \geqslant 6] < 1/10.$$

7. The uniform probability space consists of permutations $(x_1, \dots, x_n)$ of numbers from 1 to n. Find the expectation of numbers that didn't change their places. Formally the random variable is a cardinality of the set $\{i \mid x_i = i\}$.