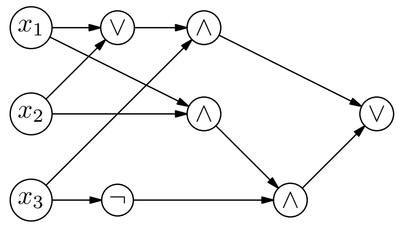
Discrete Mathematics **DSBA**

Seminar 18. Boolean circuits. Functional completeness

1. Describe a boolean function defined by the boolean circuit (on the picture)



Zhegalkin polynomial is a formula of a form

$$\bigoplus_{S\subseteq\{1,\ldots,n\}} a_S \bigwedge_{i\in S} x_i, \qquad a_S\in\{0,1\},$$

The boolean values a_S are the coefficients of the Zhegalkin polynomial.

2. Is it true that any boolean function can be represented by Zhegalkin polynomial?

3. Do there exist a boolean function f on two variables such that the boolean circuit with gates $\{\wedge, f\}$

$$x_1,\ x_2,\ s_1 := f(x_1,x_2);\ s_2 := f(x_2,x_1);\ s_3 := s_1\wedge s_2$$

computes **a)** function x_1? **b)** function $x_1\oplus x_2$?

4. Verify whether the sets of logic connectivities are functional complete

a) $\{\neg,\to\}$

b) $\{\wedge,\vee,\setminus\}$, where $x\setminus y$ equals to $x\wedge\neg y$?

c) $\{1,\oplus\}$

d) $\{\neg;\equiv\}$, where $x\equiv y$ equals to $(x\to y)\wedge(y\to x)$?

5. Majority is a boolean function $\mathrm{MAJ}(x_1,x_2,\ldots,x_n)$ that returns the most frequent value among $x_1,\ldots,x_n$. Assume that if the numbers of zeroes and ones are the same then Majority returns 0. Circuits with gates $\{\vee,\wedge,1,0\}$ are called monotone circuits. Is MAJ computable via monotone circuits?

6. A boolean function $f(x_1,\ldots,x_n)$ is self-dual if for all values $a_1,\ldots,a_n$ the following equality holds

$$f(a_1,\ldots,a_n)=\neg f(\neg a_1,\ldots,\neg a_n).$$

a) Are the functions $x_1\vee x_2$, $x_1\wedge x_2$ self-dual?

b) Prove that circuits that have all self-dual gates compute self-dual functions.

7. Prove that any boolean circuit of size s with n variables can be converted to a boolean circuit in which all negations apply only to variables, and at the same time the size of the new scheme does not exceed $p(s,n)$, where p is a fixed polynomial.

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Home Assignment 18

1. Can you compute a zero-function ($f(x_1, \dots, x_n) = 0$ for all values of x_is) via circuits with gates $\neg(x_1 \to x_2)$?

2. Construct a circuit with gates $\{1, \wedge, x_1 \oplus x_2\}$ (Zhegalkin basis) that computes the function $\mathrm{MAJ}(x, y, z)$. (See the definition of MAJ on the previous page.)

3. The function f is computable by the following circuit with gates $\{\neg\mathrm{MAJ}(x_1, x_2, x_3), \mathrm{MAJ}(x_1, x_2, x_3)\}$:

$$x_1,\ x_2,\ x_3,\ s_1 := \mathrm{MAJ}(x_1, x_2, x_3);\ s_2 := \neg\mathrm{MAJ}(x_1, x_2, x_3);\ s_3 := \neg\mathrm{MAJ}(s_1, s_2, s_1).$$

Construct a circuit with smaller number of assignments that computes f (simplify the given circuit).

4. Prove that the set of logical connectivities consisting of the only function $x \mid y = \neg(x \wedge y)$ is functionally complete. This function is known as NAND.

5. Is the set $\{\vee; \to\}$ functionally complete?

6. Is the set $\{\neg, \mathrm{MAJ}(x_1, x_2, x_3)\}$ functionally complete?

7. Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone* if for all pairs $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \{0,1\}^n$ the condition $\forall i\ x_i \leqslant y_i$ implies $f(x_1, \dots, x_n) \leqslant f(y_1, \dots, y_n)$.

Let $f(x_1, \dots, x_n)$ be a non-monotone function. Prove that the negation function ($\neg x_i$) is computable by a circuit over the basis $\{0, 1, f\}$.

8. Prove that a monotone function is computable by a monotone circuit (see the definition on the previous page.)