Discrete Mathematics **DSBA**

Seminar 19. Boolean circuits II. Circuits, Algorithms and Complexity

If the basis is not described in the problem's statement, then you shall use the common basis $\{\neg, \wedge, \vee\}$.

If a graph is mentioned in the problem, it is assumed that the circuit has $\binom{n}{2}$ inputs, each of them corresponds to an edge (if one is on the input then there is the edge in the graph).

1. Construct a polynomial circuit that computes $\mathrm{MAJ}(x_1, x_2, \ldots, x_n)$. (Recall that Majority is a boolean function $\mathrm{MAJ}(x_1, x_2, \ldots, x_n)$ that returns the most frequent value among $x_1, \ldots, x_n$. Assume that if the numbers of zeroes and ones are the same then Majority returns 0.)

2. A boolean function $L(x_1, \ldots, x_n; y_1, \ldots, y_n)$ computes which binary number on the input is greater. Formally, it returns 1 if and only if $(\overline{x_1, \ldots, x_n})_2 < (\overline{y_1, \ldots, y_n})_2$. Construct a boolean circuit of size $O(n)$ that computes $L(x_1, \ldots, x_n; y_1, \ldots, y_n)$.

3. a) Construct a polynomial boolean circuit that "sorts" bits on the input. Formally if the input is $(x_1, x_2, \ldots, x_n)$ and exactly k variables are assigned to 1 then on the output $(y_1, \ldots, y_n)$ first k bits are equal to 1 and all the rest are equal to zero. **b)** Construct the circuit with gates $\{\wedge, \vee\}$.

4. Prove that each boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is computable by a circuit of size $O(2^n)$. (Upgrade a DNF construction or design a recursive construction).

5. Construct a polynomial circuit for the function $f\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ that equals one if and only if the input graph is connected and contains the Eulerian cycle.

6. Let $n = k + 2^k$. Pointer function $f(x_1, \ldots, x_k, y_0, \ldots, y_{2^k-1})$ equals y_x, where x is a number with binary representation $x_1 \ldots x_k$. Construct a polynomial scheme for the pointer function.

7. Prove that you can arbitrary define a boolean function $f(x_1, x_2, \ldots, x_{2n})$ on all the tuples $(a_1, a_2, \ldots, a_{2n})$ with exactly n zeroes and after that you can define it on the rest tuples so that f would be a monotone boolean function.

Recall that boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone* if for all pairs $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \{0,1\}^n$ the condition $\forall i\ x_i \leqslant y_i$ implies $f(x_1, \ldots, x_n) \leqslant f(y_1, \ldots, y_n)$.

8. Prove that for all sufficiently large n there exists a monotone Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, that is not computable by any circuit of size less than n^{100}.

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Home Assignment 19

If the basis is not described in the problem's statement, then you shall use the common basis $\{\neg, \wedge, \vee\}$.

1. Construct a polynomial circuit that verifies whether an n-bit binary number on the input is divisible by 3.

2. A triangle in a graph is a triple of vertices connected to each other. Consider the input of a function $T\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ as an undirected graph on n vertices and put $T(G) = 1$ if and only if G has no triangles. Construct a polynomial circuit that computes the function T.

3. Construct a polynomial circuit for the function $f\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$, that returns 1 if and only if the input graph is two-colorable.

4. A Boolean function $W(u_1, \ldots, u_\ell; x_1, \ldots, x_n)$ returns 1 if and only if the binary word $u_1 u_2 \ldots u_\ell$ is a subword of the binary word $x_1 x_2 \ldots x_n$ i.e. for some $0 \leqslant k \leqslant n - \ell$ the equalities $x_{k+i} = u_i$ hold. Construct a polynomial circuit circuit that computes W.

5. A boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *symmetric* if it's value does not depend on the permutation of its input bits. For example $f(1,0,1,0) = f(1,1,0,0) = f(1,0,0,1)$ for a symmetric $f(x_1, x_2, x_3, x_4)$. Prove that any symmetric boolean function is computable by a polynomial circuit.

6. Prove that each boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is computable by a circuit with gates $\{\oplus, \wedge, 1\}$ of size of at most 2^{n+1}.

7. Construct a polynomial boolean circuit with gates $\{\wedge, \vee\}$ that computes $\mathrm{MAJ}(x_1, \ldots, x_n)$.