Discrete Mathematics **DSBA**

Seminar 20. Decision trees and lower bounds

1. Among the n stones there is (exactly) one radioactive. To find it you shall use a Geiger counter which checks whether a heap of stones contains a radioactive one. Find the least required number of checks.

2. Someone wrote in the cells of the chessboard the numbers from 1 up to 64 (in an unknown order, each number presents exactly ones). You can choose an arbitrary set of cells and perform a query. A response to the query is the set of all the numbers written in the cells from the set. Find the least required number of queries to learn which number is written in which cell.

3. There are n coins among which there is exactly one fake coin and balance scales. Real coins have the same mass and are heavier then the fake one. You can weight any two groups of coins per weighting. Prove that you can find a fake coin for $\log_3 n + O(1)$ weightings.

4. Prove that you shall perform at least $\log_3 n$ weightings to find a fake coin (from the previous problem).

5. Let $n = k + 2^k$. Pointer function $f(x_1, \ldots, x_k, y_0, \ldots, y_{2^k-1})$ equals y_x, where x is a number with binary representation $x_1 \ldots x_k$. Construct a polynomial circuit for the pointer function. (Use an idea of a decision tree).

6. A computation of a boolean function $f\colon \{0,1\}^n \to \{0,1\}$ in the decision tree model is organised as follows. A some input $a_1, \ldots, a_n$ of f is fixed. You can learn by a query the value a_i of a chosen variable x_i. After performing all the queries you need you shall compute $f(a_1, \ldots, a_n)$. A decision tree complexity of f is the least number of queries required to compute the value of f on an arbitrary input. Note that you can perform each next query after learning the result of the previous one.

a) Find the decision tree complexity of a function $\bigoplus_i x_i$.

b) Find the decision tree complexity of the pointer function from the previous problem.

Recall that boolean function $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone* if for all pairs $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \{0,1\}^n$ the condition $\forall i\ x_i \leqslant y_i$ implies $f(x_1, \ldots, x_n) \leqslant f(y_1, \ldots, y_n)$.

7. Prove that for all sufficiently large n there exists a monotone Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, that is not computable by any circuit of size less than n^{100}.