Discrete Mathematics **DSBA**

Zero Variant of the Spring Exam

Preambula. Recall that answers without arguments are not considered as solutions. You shall use convincing arguments so, that your solution becomes close to a mathematical proof.

1. Find the maximum possible number of edges in a directed acyclic graph on n vertices. (Recall that there are no parallel edges.)

2. An undirected graph-path consists of vertices $v_0, v_1, \ldots, v_4$ (each subsequent vertices are adjacent). The vertices are uniformly randomly painted in 4 colors. Find the probability of that each pair of vertices (v_i, v_j) such that the distance between v_i and v_j is 1 or 2 is painted in different colors.

3. Find $\gcd(\underbrace{11\ldots11}_{120\text{ times}}, \underbrace{11\ldots11}_{84\text{ times}})$. (Numbers written in decimal representation.)

4. Someone has chosen $2^{n-1}+1$ subsets of an n-element set. Prove that there are two non-intersecting subsets among the chosen subsets.

5. A boolean function $U_2(x_1, \ldots, x_n)$ equals one if and only if among the input bits $x_1, \ldots, x_n$ there are exactly two ones. Construct a circuit of size $O(n)$ that computes U_2.

6. Let X and Y be finite sets and $f, g\colon X \to Y$ are totally defined functions. It is known that f is an injection and g is a surjection. Does it imply that for each subset $A \subseteq X$ the assertion $|g^{-1}(f(A))| \geqslant |A|$ holds? If you answer is true, provide the proof and otherwise provide a counterexample (f, g and A for which the assertion doesn't hold).

7. There are 6 black and 6 white pearls of the same form. A jeweller creates a random necklace by putting them on the thread in (uniformly) random order and after that the ends of the thread are tied and all the pearls are arranged in a circle. Find the expectation of black pearls that have both white neighbors.

8. A set $\mathbb{R}$ of real numbers is splited into two subsets A and B, i.e. $A \cup B = \mathbb{R}$, $A \cap B = \varnothing$. Prove that at least one of the subsets A or B has cardinality continuum.